Scaling properties of growing noninfinitesimal perturbations in space-time chaos

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(Received 9 February 2004; published 29 November 2004)

We study the spatiotemporal dynamics of random spatially distributed noninfinitesimal perturbations in one-dimensional chaotic extended systems. We find that an initial perturbation of finite size ϵ_0 grows in time obeying the tangent space dynamic equations (Lyapunov vectors) up to a characteristic time $t_\times(\epsilon_0) \sim b - (1/\lambda_{max})\ln(\epsilon_0)$, where λ_{max} is the largest Lyapunov exponent and b is a constant. For times $t < t_\times$, perturbations exhibit spatial correlations up to a typical distance $\xi \sim t^z$. For times larger than $t_\times$, finite perturbations are no longer described by tangent space equations, memory of spatial correlations is progressively destroyed, and perturbations become spatiotemporal white noise. We are able to explain these results by mapping the problem to the Kardar-Parisi-Zhang universality class of surface growth.

DOI: 10.1103/PhysRevE.70.056224 PACS number(s): 05.45.Jn, 05.45.Ra, 05.40.−a

I. INTRODUCTION

A standard tool for studying chaotic behavior in dynamical systems is the computation of the characteristic Lyapunov exponents, which measure the typical exponential growth rate of an infinitesimal disturbance [1,2]. The characteristic Lyapunov exponents in extended systems are defined in a similar way to their low-dimensional counterpart and can be calculated from the linearization of the equations of motion [3,4]. The main point is that the growth of an infinitesimal perturbation is described by the linear equations for the tangent space, the so-called Lyapunov vectors (see below). However, for many practical purposes the dynamics of infinitesimal perturbations may be irrelevant as indicators of, for instance, the predictability time. Indeed, in many realistic situations the error in the initial conditions is finite. The important fact is that the evolution of finite errors is not confined to the tangent space, as defined by the growth of linearized perturbations, but is controlled by the complete nonlinear dynamics. A good example with important practical application occurs in weather forecasting, where one deals with the whole Earth's atmosphere—an extremely high-dimensional system in which initial conditions can be determined only with limited accuracy. The effects of finite perturbations have recently been studied in the context of fully developed turbulence [5]. In order to deal with realistic perturbations, the concept of finite-size Lyapunov exponents has been found to be useful to analyze predictability in high-dimensional chaotic systems [4,5].

In this paper, we study the dynamics of random spatially distributed finite-size errors in chaotic extended systems and focus on their propagation dynamics. We argue that, after a suitable transformation of variables, the dynamics of finite perturbations can be interpreted as a kinetic roughening process in the Kardar-Parisi-Zhang (KPZ) universality class [10]. We find that, due to the finiteness of the initial error, there is a characteristic time scale $t_\times(\epsilon_0) \sim b - (1/\lambda_{max})\ln(\epsilon_0)$, where λ_{max} is the largest Lyapunov exponent, ϵ_0 is a measure of the initial size of the perturbation, and b is a constant. For times $t < t_\times$, the dynamic evolution of a finite perturbation is governed by the Lyapunov vector. In this regime, finite perturbations become spatially correlated up to a typical length scale $\xi \sim t_\times^z$, where z is the dynamic exponent of the KPZ problem ($z=3/2$ for one-dimensional systems). However, for times $t > t_\times$, any finite perturbation leaves the tangent space and is no longer described by the Lyapunov vectors. In this late regime, memory of spatial correlations is progressively destroyed and perturbations actually become white noise in space and time. Our approach provides new tools for studying chaotic extended systems by allowing us to fully describe the spread of correlations, to estimate the spatial extent of correlations of the chaotic field, and to measure the effective number of degrees of freedom.

We exemplify our results by means of numerical simulations of coupled map lattices in one dimension, which are simple model systems exhibiting space-time chaos and convenient as far as the computing time is concerned. We consider a coupled map array consisting of L chaotic oscillators given by

$$u(x,t+1) = \nu f(u(x+1,t)) + \nu f(u(x-1,t)) + (1-2\nu) f(u(x,t)), \tag{1}$$

where $x=1,2,\ldots,L$, $f(u)$ is a chaotic map, ν is the coupling constant, and periodic boundary conditions are imposed. We have fixed the coupling to $\nu=1/3$ in all the simulations presented in this paper. We have carried out simulations for two different choices of the map, the chaotic logistic map $f(u) = 4u(1-u)$, $0 \leqslant u \leqslant 1$ and the tent map $f(u) = 1-2|u-1/2|$, $0 \leqslant u \leqslant 1$. For the sake of brevity, all the results we present below correspond to coupled logistic maps, but similar results were obtained for the tent map.

The dynamics of *infinitesimal* perturbations of a turbulent

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state $u^0(x,t)$ can be studied by linearizing the evolution equation (1). This leads to the tangent space equations or Lyapunov vector $\delta u(x,t)$ evolution equation

$$\delta u(x,t+1) = \nu f'[u(x+1,t)]\delta u(x+1,t) + \nu f'[u(x-1,t)]\delta u(x-1,t) + (1-2\nu)f'[u(x,t)]\delta u(x,t) + O[(\delta u)^2], \quad (2)$$

where $f'[u(x,t)] = df(y)/dy|_{u(x,t)}$. This implies to solve simultaneously the field $u(x,t)$ evolution equation (1). The Lyapunov vector is defined in the linear approximation, when higher-order corrections $O[(\delta u)^2]$ are neglected.

The analysis of infinitesimal perturbations allows us to compute several indicators characterizing the chaotic system, including the whole spectrum of Lyapunov exponents, and to investigate how this depends on system size [3,4,6]. However, as explained above, there are indeed many situations where the Lyapunov analysis has no relevance due, for instance, to the finite nature of the initial errors.

II. SCALING OF FINITE-SIZE PERTURBATIONS

Let us now consider the evolution of random finite-size perturbed trajectories in our model system (1). Given an initial condition $u^0(x,0)$, the solution $u^0(x,t)$ is determined by computing Eq. (1) for a number t of time steps. This is our reference trajectory and we are interested in the evolution of finite perturbations around that reference solution. For reasons that will become clear below, we find it convenient to introduce now what we call the *amplitude factor* $\epsilon(t)$ as the spatial geometrical mean value of a perturbation,

$$\epsilon(t) \equiv \prod_{x=1}^{L} |\delta u(x,t)|^{1/L}. \quad (3)$$

As we shall see below, the amplitude factor turns out to be a very important quantity which contains the information about the dominant exponential growth rate.

Since we are interested here in the propagation of real (noninfinitesimal) errors, we should avoid linearization of Eq. (1). Instead, we compute the trajectories generated by iterating Eq. (1) for an ensemble of initial conditions $u(x,0)=u^0(x,0)+\delta u(x,0)$, where $\delta u(x,0)$ is a random finite perturbation of initial amplitude $\epsilon(t=0)=\epsilon_0$. For each iteration of the lattice (1), the difference $\delta u(x,t)=u(x,t)-u^0(x,t)$ between the reference trajectory and every one of the perturbed solutions is calculated. Although the disturbances are initially random and uncorrelated in space, as time goes by, they grow and get spatially correlated. The statistical fluctuations of errors can be characterized by studying the ensemble of finite perturbations $\{\delta u_n(x,t)\}_{n=1}^{N}$, which correspond to N independent realizations of the initial perturbation.

In Fig. 1 (upper panel), we plot $\ln\langle\epsilon(t)\rangle$ versus time for different values of the initial perturbation amplitude ϵ_0, where $\langle\cdots\rangle$ stands for average over realizations of the initial perturbation. One can immediately see that there exists a characteristic time scale $t_\times(\epsilon_0)$ such that for times $t < t_\times(\epsilon_0)$, the average amplitude factor grows exponentially in time $\epsilon(t) \approx \epsilon_0 \exp(\lambda t)$, where $\lambda = 0.343 \pm 0.005$. We demonstrate below that λ indeed corresponds to the maximal Lyapunov exponent. For longer times, $t > t_\times(\epsilon_0)$, the amplitude factor saturates to a constant value. Both the saturation constant and the maximal Lyapunov exponent are independent of the initial perturbation size ϵ_0. However, the saturation times $t_\times(\epsilon_0)$ increase as the size of the initial perturbation ϵ_0 becomes smaller. The characteristic time scale $t_\times$ corresponds to the crossover time at which the dynamics of a finite-size perturbation depart from the linear evolution (i.e., the Lyapunov vectors) given by Eq. (2). This crossover occurs because Lyapunov vectors describe only the behavior of strictly infinitesimal perturbations. One can estimate $t_\times$ roughly as the time at which $\epsilon(t)$ reaches some δ, so that the higher-order terms $O[(\delta u)^2]$ cannot be neglected in the evolution equation (2). This crossover takes place at a typical time $t_\times(\epsilon_0) \sim (1/\lambda)\ln(\delta) - (1/\lambda)\ln(\epsilon_0)$. Therefore, for times $t > t_\times(\epsilon_0)$, nonlinear corrections, due to finiteness of the initial perturbation, come into play and drive errors out of the tangent space. From then on, the linear approximation cannot describe the evolution of errors. One then expects that $t_\times(\epsilon_0) \to \infty$ as $\epsilon_0 \to 0$.

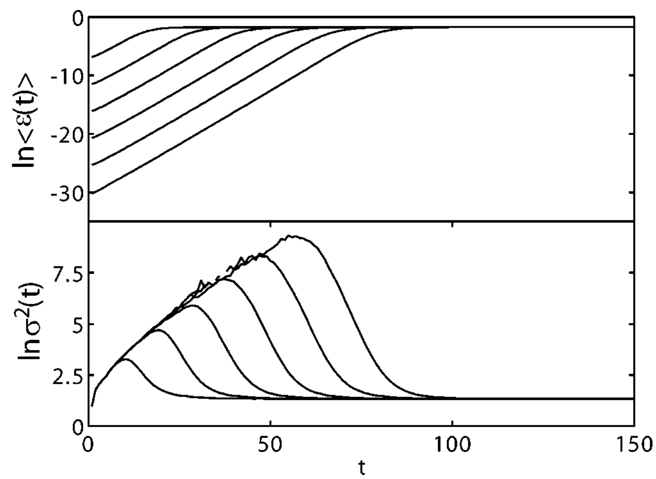

FIG. 1. Numerical results for the propagation of finite-size errors in coupled logistic maps. Upper panel shows the ensemble averaged amplitude factor $\langle\epsilon(t)\rangle$ vs time for perturbations starting with initial amplitudes of $\epsilon_0 = 10^{-3}$, 10^{-5}, 10^{-7}, 10^{-9}, 10^{-11}, and 10^{-13} (from top to bottom) in 1D lattices of $L=1024$ sites. Results were averaged over 600 different initial conditions. Lower panel shows the variance $\sigma^2(t)$ for the same initial perturbations as before and ϵ_0 decreasing from 10^{-3} (leftmost curve) to 10^{-13} (rightmost curve).

Besides exponential growth, spatial correlations are dynamically generated during the evolution of the perturbation. Correlations contain information about the subleading Lyapunov exponents and thus also contribute to the perturbation size growth. The important role of correlations can be better realized after subtraction of the dominant exponential growth component given by $\epsilon(t)$. We find that a very useful indicator is given by the *reduced* perturbations $\delta r(x,t)$ that we define as

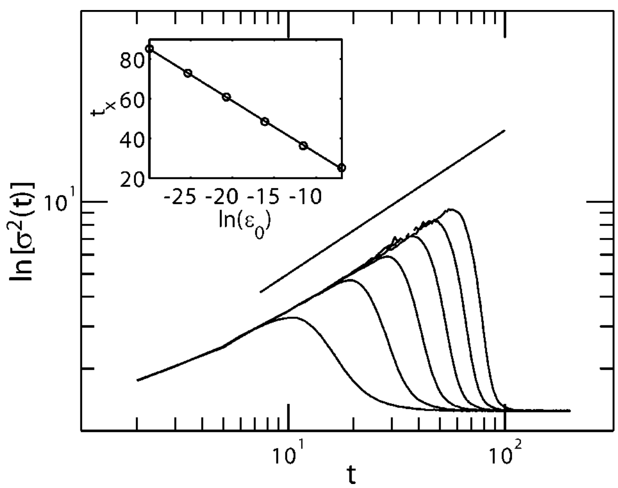

FIG. 2. The same numerical results for the variance $\sigma^2(t)$ as those in Fig. 1 are plotted in the main panel in log-log scale to show the power-law behavior $\ln \sigma^2(t) \sim t^{2\beta}$. The straight line has a slope 0.60 and is plotted to guide the eye. The inset shows the mean-square fit of the characteristic time scale $t_\times(\epsilon_0)$ obtained from the saturation times of the amplitude factor $\epsilon(t)$. The slope of the straight line is $1/\lambda = 2.86(5)$.

$$\delta r(x,t) = \frac{\delta u(x,t)}{\epsilon(t)}, \tag{4}$$

where the dominant exponential growth is globally removed and one is left with the effect of correlations. For random initial errors, i.e., random spatially distributed initial conditions, the average $\langle \overline{\delta r(x,t)} \rangle$ vanishes, where $\langle \cdots \rangle$ stands for the average over realizations of the initial perturbation and the overbar is a spatial average. Thus, statistical fluctuations of the reduced perturbations are measured by the variance $\sigma^2(t) = \langle \overline{\delta r(x,t)^2} \rangle$. As we shall see below, $\sigma(t)$ gives information about the growth of perturbations due solely to correlations.

In Fig. 1 (lower panel), we show our numerical results for the variance of the reduced perturbations, Eq. (4). The variance $\sigma^2(t)$ grows as $\sigma^2(t) \sim \exp(t^{2\beta})$ for times $t < t_\times$, i.e., before saturation occurs due to finiteness of the initial disturbance. This scaling behavior is better seen in Fig. 2, where we plot in log-log scale $\ln(\sigma^2)$ vs time and the exponent $\beta = 0.30 \pm 0.05$ (see Fig. 2). As before, this rapid growth occurs for times $t < t_\times$ when the dynamics of disturbances are well described by the Lyapunov vectors. One can determine the crossover time $t_\times$ from Fig. 1, either by measuring the saturation times of $\epsilon(t)$ (upper panel) or by measuring the time shift of the maxima of $\sigma^2(t)$ (lower panel). Results are shown in the inset of Fig. 2, where a straight line fit of the data leads to $t_\times(\epsilon_0) = b - (1/\lambda)\ln(\epsilon_0)$ with b a constant. The slope of the fit $1/\lambda = 2.86 \pm 0.05$ is in excellent agreement with our previous determination of $\lambda \approx 0.34$ from the exponential growth of the amplitude factor $\epsilon(t) \sim \exp(\lambda t)$.

The magnitude and extent of spatial correlations can be measured by using the site-site correlation function $G(l,t) = \langle \overline{\delta r(x_0,t)\delta r(x_0+l,t)} \rangle / \langle \overline{\delta r(x_0,t)^2} \rangle$. The magnitude of spatial correlations increases in time $G(l,t_1) < G(l,t_2)$ if $t_1 < t_2$ for times $t_1, t_2 < t_\times$. However, as shown in Fig. 3, correlations become progressively smaller for times $t > t_\times$. This again demonstrates that the effect of having initially finite perturbations is to introduce a characteristic time $t_\times(\epsilon_0)$ marking the typical time it takes for the system to depart from tangent space (with the building-up of correlations) to nonlinear evolution (uncorrelated errors). For times larger than $t_\times$, the perturbation spatial correlations are progressively destroyed, as can be seen from the decay of the correlations $G(l,t)$ in Fig. 3. Further Fourier spectrum analysis [11] shows that errors actually become indistinguishable from white noise.

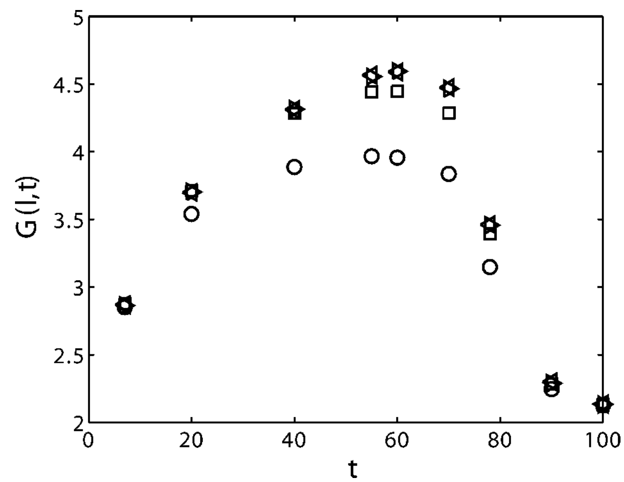

FIG. 3. Time behavior of the correlation function for lattice sites separated at different distances $l=8$ (○), $l=16$ (□), $l=64$ (◁), and $l=185$ (▷) in an array of $L=1024$ coupled maps.

III. KINETIC ROUGHENING PICTURE

In order to make a theoretical interpretation of our numerical results, in particular the existence of scaling behavior, we can resort to the Hopf-Cole transformation, first proposed by Pikovsky and Politi for the actual Lyapunov vectors [12,13], that allows us to link spreading of errors to nonequilibrium surface growth. We find that the dynamics of finite perturbations can also be seen as a kinetic roughening process of the surface defined by $h(x,t) = \ln|\delta u(x,t)|$. This is a very useful transformation that allows us to borrow some well-known results in the field of nonequilibrium surface growth. Indeed, Politi and Pikovsky [12,13] have shown that errors in many extended systems lead to a surface growth process in the universality class of KPZ [10].

Let us first consider the amplitude factor defined in Eq. (3) and show that it is related to the average surface velocity. We can write $|\delta u(x,t)| = \exp[h(x,t)]$ and thus $\epsilon(t) = \exp[(1/L)\sum_{x=1}^{x=L} h(x,t)] = \exp[\overline{h}(t)]$. We then obtain that the amplitude factor must grow as $\epsilon(t) \sim \epsilon_0 \exp(\lambda_{\max} t)$, where $\lambda_{\max}$ is the largest Lyapunov exponent and corresponds in this mapping to the surface velocity, $\overline{h}(t) = \lambda_{\max} t + \overline{h}(0)$ [12,13]. Direct measures of the average surface velocity (not shown) are in excellent agreement with our above-discussed estimations of λ. As an independent check, we have also measured the largest Lyapunov exponent by standard techniques [14,15].

Also the time behavior of the variance of the reduced perturbations shown in Fig. 2 can be linked to surface scaling properties. From the surface definition $h(x,t)=\ln|\delta u(x,t)|$ we have $\sigma^2(t,L)=\langle\overline{\exp[2y(x,t)]}\rangle$, where $y(x,t)=h(x,t)-\overline{h}(t)$. We can now proceed and calculate σ^2 explicitly by considering the cumulant generating function of the stochastic variable y given by $\Phi(s)=\ln\langle\overline{\exp(isy)}\rangle$ [16], where i is the imaginary unit. This generating function can be expanded in a power series of the argument s and then evaluated at $s=-2i$ to obtain an exact expression for the variance σ^2,

$$\sigma^2(t,L) = \exp\left(\sum_{r=1}^{\infty} 2^r \frac{\langle\langle\overline{y^r}\rangle\rangle}{r!}\right), \tag{5}$$

where $\langle\langle\overline{y^r}\rangle\rangle$ are the cumulants of y [16]. In general, no closed simple formula exists for the cumulant of order r of a given variable y, but the first cumulants are given by $\langle\langle\overline{y}\rangle\rangle=\langle\overline{y}\rangle$, $\langle\langle\overline{y^2}\rangle\rangle=\langle\overline{y^2}\rangle-\langle\overline{y}\rangle^2$, and $\langle\langle\overline{y^3}\rangle\rangle=\langle\overline{y^3}\rangle-2\langle\overline{y^2}\rangle\langle\overline{y}\rangle+2\langle\overline{y}\rangle^3$, etc. [16]. In our case, $\langle\overline{y}\rangle=0$ and the second cumulant $\langle\langle\overline{y^2}\rangle\rangle=W^2(t,L)$ corresponds to the surface width $W^2(t,L)=\langle\overline{[h(x,t)-\overline{h}]^2}\rangle$. To leading order, we have

$$\sigma^2(t,L) \approx \exp[2W^2(t,L)]. \tag{6}$$

Higher-order cumulants are also finite, but their contribution to the series gets smaller as r increases. We have checked numerically that including the third and fourth cumulants shifts the exponent by less than 10%. Therefore, within this somewhat crude approximation, Eq. (6), we expect to have an estimation of the *effective* scaling exponent of $\sigma^2\sim\exp(t^{2\beta})$ with less than a 10% error bar. A clarification is now in order. For systems in which the largest Lyapunov exponent $\lambda\to 0$, one would expect to have larger crossover times $t_\times$, so that higher-order cumulants may eventually become relevant, yielding corrections to Eq. (6). In this sense, the scaling picture has to be considered as approximate. However, the scaling approach is consistent for hyperchaotic systems, such as those we are interested in here, in which not only is the leading Lyapunov exponent finite, but a series of finite positive Lyapunov exponents exists whose number increases with the system size.

The ansatz of a KPZ behavior for the surface h then implies that the width should scale as $W(t,L)\sim t^{\beta}$ for times $t<t_s(L)$ and saturates to a size-dependent value, $W(t,L)\sim L^{\alpha}$ for $t>t_s(L)$, where β and α are the growth and roughness exponent, respectively, that are known to take the values $\beta=1/3$ and $\alpha=1/2$ for the KPZ universality class in one dimension [10,17]. This is consistent with the approximate $\exp(t^{2/3})$ scaling observed in Fig. 2, bearing in mind the limitation of the approximation (6). It is worth mentioning that all the numerical results presented here are obtained for system sizes such that $t_\times \ll t_s$, so that the existence of $t_\times$ could be clearly observed.

IV. CONCLUSIONS

We have studied the spatiotemporal dynamics of finite-size perturbations in extended systems exhibiting chaos. We have introduced what we call the amplitude factor $\epsilon(t)$ in Eq. (3) and showed that this quantity is directly related to the maximal Lyapunov exponent and exhibits nice scaling properties. The amplitude factor allows us to remove in a simple and self-consistent way the dominant growth component of the perturbation at every time step, so that spatial correlations can be determined more easily. At variance with infinitesimal perturbations, finite-size errors lead to a characteristic time scale $t_\times(\epsilon_0)$ signaling the departure of the finite perturbation from the linear approach. For times $t>t_\times$, the evolution is nonlinear and spatial correlations are destroyed. We have found that the perturbation variance grows as $\sigma(t)\sim\exp(t^{2/3})$, once the contribution of the largest growth rate is removed. We have explained this behavior by representing the error growth as a kinetically rough 1D KPZ surface.

The interpretation of the exponentially growing errors $\delta u(x,t)$ as a roughening surface $h(x,t)=\ln|\delta u(x,t)|$ has many advantages since it allows us to translate the problem of analyzing deterministic chaotic fluctuations into the simpler framework of kinetic roughening of the associated surface $h(x,t)$. The scale-invariant character of the surface fluctuations leads to simple power-law behavior of the relevant quantities that can be characterized by a few critical exponents. As a byproduct, this leads in a rather simple way to an estimation of the spatial extent of correlations at time t as $\xi\sim t^z$, where $z=3/2$ is the dynamic exponent of the KPZ problem. Further results of practical application, including the relevance for the predictability problem of the propagation of finite-size errors and bred vectors, as used in the context of weather forecasting [7–9], will be published elsewhere [11].

ACKNOWLEDGMENTS

This work was supported by the CICYT (Spain) through Grants No. BFM2000-0628-C03-02 and No. BFM2003-07749-C05-03 as well as the EU Commission through Grant No. OCCULT IST-2000-29683.

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